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Office 519

Linear algebra I

Fall Semester 2013

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Problem Set #1

Due Tuesday 11 February in class

In the following K is a field.

Exercise 1 ($\star\star$):
Let $V = \{(a_1, a_2) : a_1, a_2 \in K\}$. For (a_1, a_2), $(b_1, b_2) \in V$ and $\alpha \in K$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

and

$$\alpha \cdot (c_1, c_2) = \begin{cases} (0,0) & \text{if } \alpha = 0 \\ \left(\alpha a_1, \frac{a_2}{\alpha}\right) & \text{if } \alpha \neq 0 \end{cases}$$

Is V a vector space under these operations? Justify your answer.

Exercise 2 ($\star$): Recall that the set of the subspace of some vector space is inductively ordered by the inclusion, in the sense that if W_1 and W_2 are two subspace of V then there exist a subspace W_M of V containing W_1 and W_2 and one subspace W_m of V contained in W_1 and W_2. More generally, for a arbitrary family of subspace $\{W_\lambda\}_{\lambda \in I}$, $\cap_{\lambda \in I} W_\lambda$ is a subspace and the smallest subspace contained in all W_λ in the sense if some subspace M contains all W_λ, then $\cap_{\lambda \in I} W_\lambda \subseteq M$. Here, we describe W_M.
If W_1, W_2 are subspace in a vector space V, prove that

1. $W_1 + W_2 = \{a + b : a \in W_1, b \in W_2\}$ is a vector space,
2. $W_1 + W_2$ contains both W_1 and W_2,
3. $W_1 + W_2$ is the smallest vector subspace $M \subseteq V$ that contains both W_1 and W_2,
4. $W_1 + W_2 = K - span(W_1 \cup W_2)$.
5. If $W_1 \cap W_2 = \{0\}$, we write $W_1 \oplus W_2$ instead of $W_1 + W_2$ and it is called the direct sum of W_1 and W_2. Prove that under this assumption, any element of $W_1 \oplus W_2$ can be written **uniquely** into a sum $w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$.

Exercise 3 ($\star\star$):
Suppose that $K \neq \mathbb{Z}/2\mathbb{Z}$. Recall that a direct sum between two subspace W_1 and W_2 of some vector space V is a subspace of V denote $W_1 \oplus W_2$ defined if $W_1 \cap W_2 = \{0\}$ and

$$W_1 \oplus W_2 = \{w_1 + w_2 | w_1 \in W_1 \text{ and } w_2 \in W_2\}$$

. Let $W_1 = \{A \in M_{n \times n}(K) : A_{i,j} = 0 \text{ whenever } i \leq j\}$ and let W_2 denote the set of symetric $n \times n$ matrices. Both W_1 and W_2 are subspaces of $M_{n \times n}(K)$. Prove that

$M_{n\times n} = W_1 \oplus W_2$.
Hint : Try substituting $A = U + S$ in the formulas for the symmetric and antisymmetric parts of A $A_s = \frac{1}{2}(A + A^t)$ and $A_a = \frac{1}{2}(A - A^t)$ (recall, $A = A_s + A_a$). The S in $A = U + S$ is nor the same as A_s.

Exercise 4 ($\star$):
Consider the following vectors in $\mathbb{R}^3$.

$$u_1 = (1,2,3)^t \quad u_2 = (2,3,4)^t \quad u_3 = (2,7,12)^t$$

Are these vectors linearly independent ? Do they span all $\mathbb{R}^3$? Describe their linear span $W = K - Span(u_1, u_2, u_3)$ by exhibiting a set of basis vectors. Does $b = (2,-1,1)$ lie in the subspace W?

Exercise 5 ($\star$):
Do the following vectors form a basis for $\mathbb{R}^3$?
$u_1 = (1,2,3)$, $u_2 = (2,3,4)$ and $u_3 = (0,2,2)$. [1]

[1] $(\star)$ = easy , $(\star\star)$= medium, $(\star\star\star)$= challenge